# Review of ODEs

"ODE" or just "DE" for short

An ordinary differential equation is an equation for $y = y(x)$ which involves the derivatives of $y$.

General form for first-order (ie only involves $\frac{dy}{dx}$ and not $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc) DE:

$$\boxed{A(x,y)\,dx + B(x,y)\,dy = 0}$$

Note: you can also write this as:

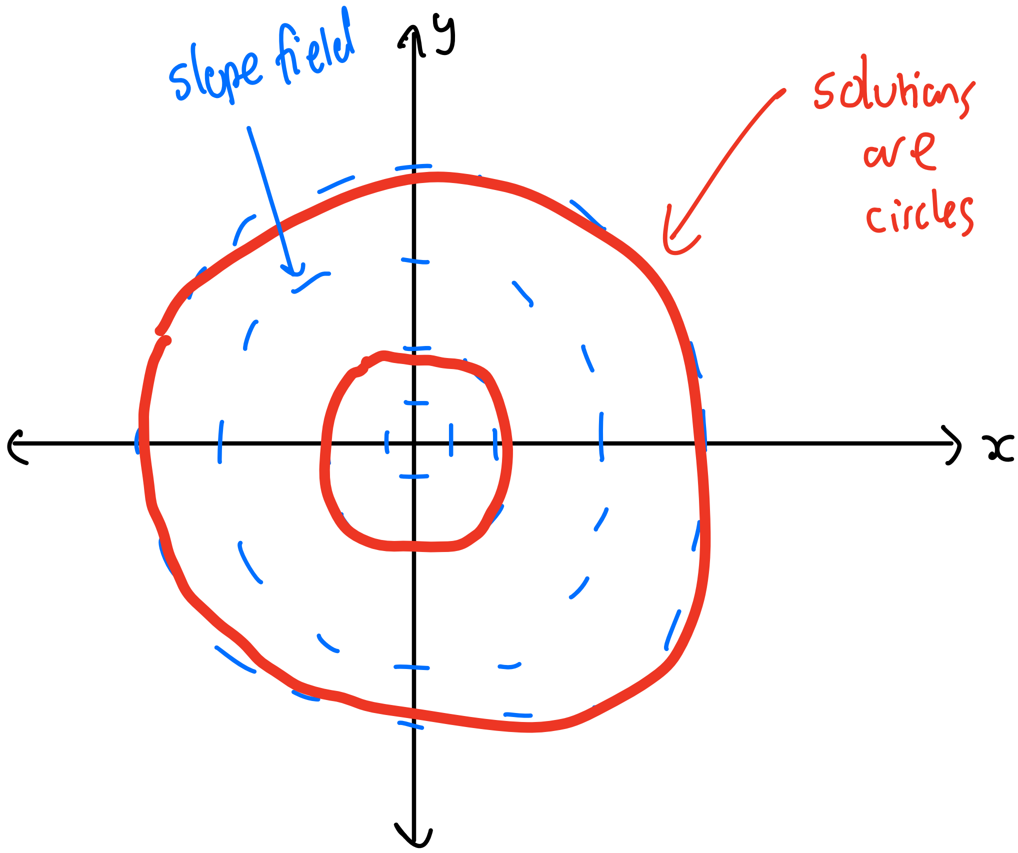
$$\boxed{\frac{dy}{dx} = -\frac{A(x,y)}{B(x,y)}} \quad (*)$$

The way to visualize a DE is that at each point $(x,y)$ in the plane the formula $(*)$ gives the slope of a solution $y(x)$ that passes through that point. This is called the slope field of the DE.

**Example** Sketch the slope field of

$$x\,dx + y\,dy = 0$$

and use it to sketch solutions of the DE.



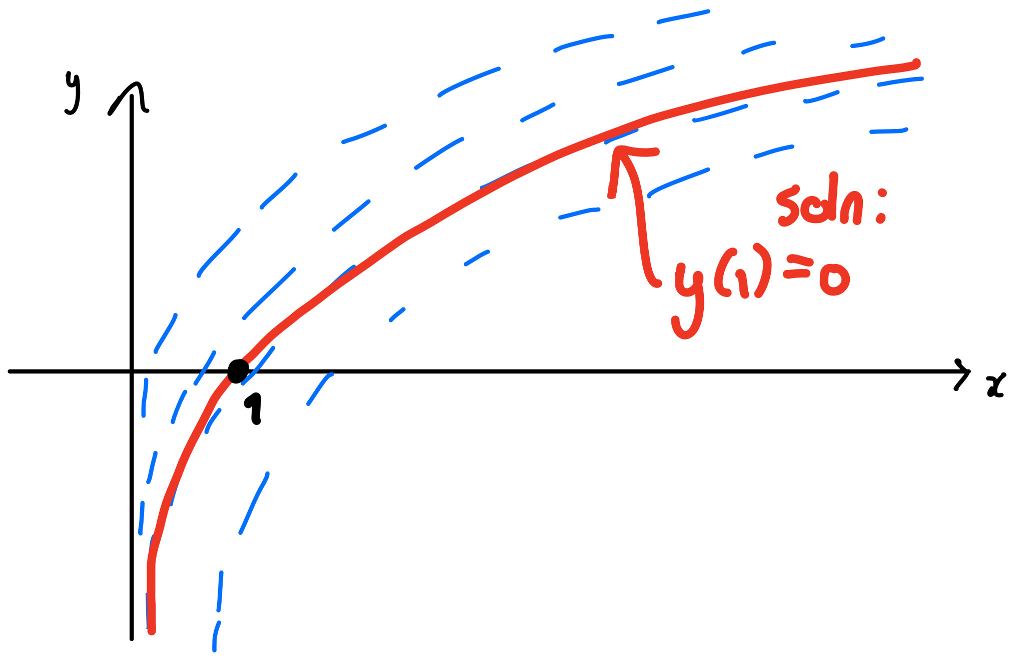
Note: Don't be fooled by the fact that we are symbolically solving lots of DE's in this course (i.e. finding explicit formulas for their solutions in terms of standard functions like polynomials, trig functions, exponentials, etc.). Most DE's can't be solved in terms of standard functions!! But, you can always <u>numerically</u> solve them (i.e. approximate their solution on a computer)... you just won't have a nice <u>formula</u> for the solution.

In fact, in practice, if a certain DE is important to us, and we don't yet have a formula for its solution in terms of our standard toolbox of functions, then we can just <u>add</u> its solution to our toolbox of functions! (This is precisely how we defined some functions in our toolbox already...)

Put yourself in the position where you know about polynomials and trig functions, and then you encounter the following DE:

Example  Plot the slope field and sketch an approximate solution to:

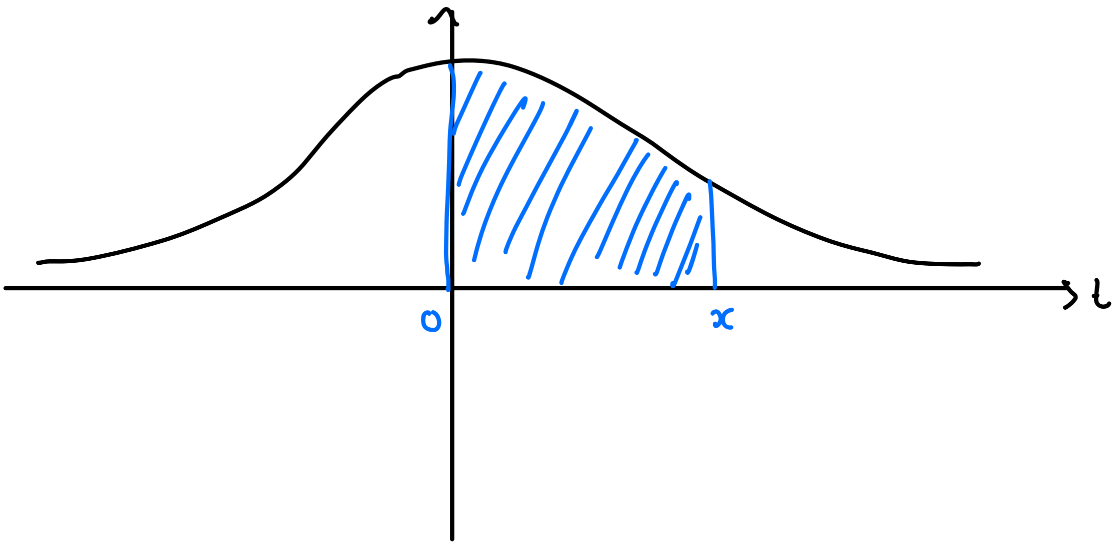
$$\frac{dy}{dx} = \frac{1}{x} \qquad , \quad y(1) = 0.$$




$$\boxed{y = \int_0^x \frac{1}{t}\, dt} \qquad \left[ = \ln x \right]$$

Example Try to solve the DE:

$$\frac{dy}{dx} = e^{-x^2} \qquad , \quad y(0) = 0$$




$$y = \int_0^x e^{-t^2}\,dt \qquad \left[ = \operatorname{erf}(x) \right]$$

**Good news ①** I have made a brilliant differential equations app, which you should always use on each problem to confirm what's going on!

(Demo)

**Good news ②** You can ask WolframAlpha (or Mathematica) to try to symbolically solve a DE for you. It is not able to write the solution $y$ as an implicit function of $x$, eg.

$$x\,dx + y\,dy = 1 \quad , \quad y(0) = 1$$

⟶ solution: $x^2 + y^2 = 1$ ← Simple

WolframAlpha / Mathematica writes $y$ explicitly as a function of $x$:

$$y(x) = \sqrt{1 - x^2}$$ ← complicated

For this reason, I recommend using SageMath (I put a link on SunLearn!) to solve DE's symbolically to check your solutions.

# Review of ODEs

## Linear DE's

$$\frac{dy}{dx} + P(x)y = f(x)$$

Method of solution: multiply both sides by integrating factor

$$I(x) = e^{\int P(x)\,dx}$$

and then recognize the LHS as the derivative of something.

**Example** Solve

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

(soln: $y = x^5 e^x - x^4 e^x + Cx^4$.)

$$y' \underbrace{- \frac{4}{x}}_{P} y = x^5 e^x \qquad (*)$$

$$I = e^{\int P\,dx} = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x}$$

$$= \frac{1}{x^4}$$

Mult. both sides of ⊛ by $I$ :

$$\therefore \frac{1}{x^4} y' - 4x^{-5} y = xe^x$$

$$\therefore \frac{d}{dx}\left(x^{-4} y\right) = xe^x$$

$$\therefore x^{-4} y = \int \underbrace{x}_{u} \underbrace{e^x dx}_{dv}$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$\therefore x^{-4} y = xe^x - e^x + C.$$

<u>Example</u> (2.3 20) Solve

$$(x+2)^2 \frac{dy}{dx} = 5 - 8y - 4xy \quad , \; y(0) = 3/4$$

(soln: $y = \dfrac{5}{(x+2)^6} - \dfrac{4}{(x+2)^4}$

$$y' + \frac{(4x+8)}{(x+2)^2}y = \frac{5}{(x+2)^2}$$

$$\therefore \quad y' + \underbrace{\frac{4}{x+2}}_{P} y = \frac{5}{(x+2)^2}$$

$$\therefore \quad I = e^{\int P dx} = e^{\int 4/x+2 \; dx} = e^{4\ln(x+2)} = (x+2)^4$$

$$\therefore \; (x+2)^4 y' + 4(x+2)^3 y = 5(x+2)^2$$

$$\therefore \; \frac{d}{dx}\left( (x+2)^4 y \right) = 5(x+2)^2$$

$$\therefore \quad (x+2)^4 y = \int 5(x+2)^2 \, dx$$

$$= \frac{5}{3}(x+2)^3 + C \, .$$

$$\therefore \quad y = \frac{5}{3}(x+2)^{-1} + \frac{C}{(x+2)^4}$$

$$y(-1) = 0$$

$$\Rightarrow 0 = \frac{5}{3} + C$$

$$\therefore C = -\frac{5}{3}$$

$$\therefore \quad y = \frac{5}{3}\left[(x+2)^{-1} - \frac{1}{(x+2)^4}\right]$$